DOI: 10.2478/auom-2019-0037
An. Şt. Univ. Ovidius Constanţa **Vol. 27(3),2019, 113–135**

Depth and Stanley depth of the edge ideals of the powers of paths and cycles

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Abstract

Let k be a positive integer. We compute depth and Stanley depth of the quotient ring of the edge ideal associated to the k^{th} power of a path on n vertices. We show that both depth and Stanley depth have the same values and can be given in terms of k and n. If $n \equiv 0, k+1, k+2, \dots, 2k(\text{mod}(2k+1))$, then we give values of depth and Stanley depth of the quotient ring of the edge ideal associated to the k^{th} power of a cycle on n vertices and tight bounds otherwise, in terms of n and k. We also compute lower bounds for the Stanley depth of the edge ideals associated to the k^{th} power of a path and a cycle and prove a conjecture of Herzog for these ideals.

1 Introduction

Let K be a field and $S := K[x_1, \dots, x_n]$ the polynomial ring over K. Let M be a finitely generated $\mathbb{Z}^n$-graded S-module. A Stanley decomposition of M is a presentation of the K-vector space M as a finite direct sum $\mathcal{D} : M = \oplus_{i=1}^{s} v_i K[W_i]$, where $v_i \in M$, $W_i \subseteq \{x_1, \dots, x_n\}$, and $v_i K[W_i]$ denotes the K-subspace of M, which is generated by all elements $v_i w$, where w is a monomial in $K[W_i]$. The $\mathbb{Z}^n$-graded K-subspace $v_i K[W_i] \subset M$ is called a Stanley space of dimension $|W_i|$, if $v_i K[W_i]$ is a free $K[W_i]$-module, where $|W_i|$ denotes the cardinality of W_i. Define $\text{sdepth}(\mathcal{D}) = \min\{|W_i| : i = 1, \dots, s\}$,

Key Words: Monomial ideal, edge ideal, depth, Stanley decomposition, Stanley depth.
2010 Mathematics Subject Classification: Primary 13C15; Secondary 13F20; 05C38; 05E99.
Received: 06.11.2018
Accepted: 11.01.2019

and $\operatorname{sdepth}(M) = \max\{\operatorname{sdepth}(\mathcal{D}) : \mathcal{D}$ is a Stanley decomposition of $M\}$. The number $\operatorname{sdepth}(\mathcal{D})$ is called the Stanley depth of decomposition $\mathcal{D}$ and $\operatorname{sdepth}(M)$ is called the Stanley depth of M. Stanley conjectured in [24] that $\operatorname{sdepth}(M) \geq \operatorname{depth}(M)$ for any $\mathbb{Z}^n$-graded S-module M. This conjecture was disproved by Duval et al. in [8] as was expected due to different nature of these two invariants. However, the relation between Stanley depth and some other invariants has already been established; see [11, 12, 21, 26]. In [11], Herzog, Vladoiu and Zheng proved that the Stanley depth of M can be computed in a finite number of steps, if $M = J/I$, where $I \subset J \subset S$ are monomial ideals. But practically it is too hard to compute Stanley depth by using this method; see for instance, [2, 5, 15, 16]. For computing Stanley depth for some classes of modules we refer the reader to [14, 20, 22, 23]. In this paper we attempt to find values and reasonable bounds for depth and Stanley depth of I and S/I, where I is the edge ideal of a power of a path or a cycle. We also compare the values of $\operatorname{sdepth}(I)$ and $\operatorname{sdepth}(S/I)$ and give positive answers to the following conjecture of Herzog.

Conjecture 1.1. *[9] Let $I \subset S$ be a monomial ideal then* $\operatorname{sdepth}(I) \geq \operatorname{sdepth}(S/I)$.

The above conjecture is proved in some other cases; see [13, 16, 20, 23]. The paper is organized as follows: First two sections are devoted to introduction, definitions, notation, and discussion of some known results. In third section, we compute depth and Stanley depth of $S/I(P_n^k)$, where $I(P_n^k)$ denotes the edge ideal of the k^{th} power of a path P_n on n vertices. Let for $q \in \mathbb{Q}$, $\lceil q \rceil$ denotes the smallest integer greater than or equal to q. Then in Theorems 3.8 and 3.14 we prove that

$$\operatorname{depth}(S/I(P_n^k)) = \operatorname{sdepth}(S/I(P_n^k)) = \lceil \frac{n}{2k+1} \rceil.$$

Let $I(C_n^k)$ be the edge ideal of the k^{th} power of a cycle C_n on n vertices. In fourth section we give some lower bounds for depth and Stanley depth of $S/I(C_n^k)$; see Theorems 4.5 and 4.7. If $n \geq 2k+2$, then by Corollaries 4.6 and 4.8 we prove that if $n \equiv 0, k+1, \dots, 2k(\operatorname{mod}(2k+1))$ then $\operatorname{depth}(S/I(C_n^k)) = \operatorname{sdepth}(S/I(C_n^k)) = \lceil \frac{n}{2k+1} \rceil$. Otherwise,

$$\lceil \frac{n}{2k+1} \rceil - 1 \leq \operatorname{depth}(S/I(C_n^k)), \operatorname{sdepth}(S/I(C_n^k)) \leq \lceil \frac{n}{2k+1} \rceil.$$

Last section is devoted to Conjecture 1.1 for $I(P_n^k)$ and $I(C_n^k)$. By our Theorem 5.2 we have

$$\operatorname{sdepth}(I(P_n^k)) \geq \lceil \frac{n}{2k+1} \rceil + 1,$$

which shows that $I(P_n^k)$ satisfies Conjecture 1.1. Let $n \geq 2k+1$. Proposition 5.3 gives a lower bound for $I(C_n^k)/I(P_n^k)$ that is

$$\text{sdepth}(I(C_n^k)/I(P_n^k)) \geq \lceil \frac{n+k+1}{2k+1} \rceil.$$

Corollary 5.5 of this paper proves that $I(C_n^k)$ satisfies Conjecture 1.1.

2 Definitions and notation

Throughout this paper $\mathfrak{m}$ denotes the unique maximal graded ideal $(x_1, \dots, x_n)$ of S. We set $S_m := K[x_1, x_2, \dots, x_m]$, $\text{supp}(v) := \{i : x_i | v\}$ and $\text{supp}(I) := \{i : x_i | u, \text{ for some } u \in \mathcal{G}(I)\}$, where $\mathcal{G}(I)$ denotes the unique minimal set of monomial generators of the monomial ideal I. Let $I \subset S$ be an ideal. Then we write I instead of IS. Thus every ideal will be considered an ideal of S unless otherwise stated. Let I and J be monomial ideals of S, then for $I+J$ we write (I, J).

We review some notation and refer the reader to [3] for further details. Let G be a simple graph. For a positive integer k, the k^{th} power of graph G is another graph G^k on the same set of vertices, such that two vertices are adjacent in G^k when their distance in G is at most k. In the whole paper we label the vertices of the graph G by $1, 2, \dots, n$. We denote the set of vertices of G by $[n] := \{1, 2, \dots, n\}$ and its edge set by $E(G)$. We assume that all graphs and their powers are simple graphs. We also assume that all graphs have at least two vertices and a non-empty edge set. For a graph G, the edge ideal $I(G)$ associated to G is defined as $I(G) := (x_i x_j : \{i, j\} \in E(G))$. For $n \geq 2$, a graph G is called a path if $E(G) = \{\{i, i+1\} : i \in [n-1]\}$. A path on n vertices is denoted by P_n. For $n \geq 3$, a graph G is called a cycle if $E(G) = \{\{i, i+1\} : i \in [n-1]\} \cup \{1, n\}$. A cycle on n vertices is denoted by C_n. For $n \geq 2$, the k^{th} power of a path, denoted by P_n^k, is a graph such that for all $1 \leq i < j \leq n$, $\{i, j\} \in E(P_n^k)$ if and only if $0 < j - i \leq k$. If $n \leq k+1$, then P_n^k is a complete graph on n vertices. If $n \geq k+2$, then

$$E(P_n^k) = \cup_{i=1}^{n-k} \big\{\{i, i+1\}, \{i, i+2\}, \dots, \{i, i+k\}\big\} \cup \\ \cup_{j=n-k+1}^{n-1} \big\{\{j, j+1\}, \{j, j+2\}, \dots, \{j, n\}\big\}.$$

For $n \geq 3$, the k^{th} power of a cycle, denoted by C_n^k, is a graph such that for all vertices $1 \leq i, j \leq n$, $\{i, j\} \in E(C_n^k)$ if and only if $|j - i| \leq k$ or $|j - i| \geq n - k$. If $n \leq 2k+1$, then C_n^k is a complete graph on n vertices. If $n \geq 2k+2$, then

$$E(C_n^k) = E(P_n^k) \cup \cup_{l=1}^{k} \big\{\{l, l+n-k\}, \{l, l+n-k+1\}, \{l, l+n-k+2\}, \dots, \{l, n\}\big\}.$$

For examples of powers of paths and cycles see Figures 1 and 2.
If $n \leq k+1$, then $I(P_n^k)$ is a squarefree Veronese ideal of degree 2. If $n \geq k+2$, then

$$\mathcal{G}(I(P_n^k)) = \cup_{i=1}^{n-k}\{x_ix_{i+1}, x_ix_{i+2}, \dots, x_ix_{i+k}\}\cup \\ \cup_{j=n-k+1}^{n-1}\{x_jx_{j+1}, x_jx_{j+2}, \dots, x_jx_n\}.$$

If $n \leq 2k+1$, then $I(C_n^k)$ is a squarefree Veronese ideal of degree 2. If $n \geq 2k+2$, then

$$\mathcal{G}(I(C_n^k)) = \mathcal{G}(I(P_n^k)) \cup \cup_{l=1}^{k}\{x_lx_{l+n-k}, x_lx_{l+n-k+1}, \dots, x_lx_n\}.$$

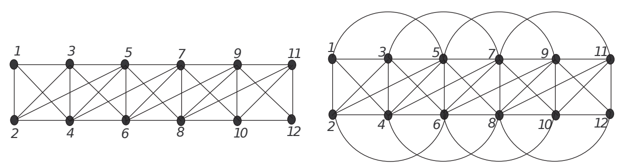

Figure 1: From left to right, P_{12}^3 and P_{12}^4 respectively.

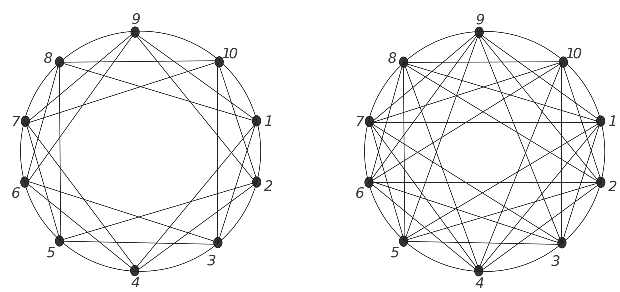

Figure 2: From left to right, C_{10}^3 and C_{10}^4 respectively.

Lemma 2.1 ([17, Lemma 3])**.** *If $n \geq k+1$, then $|\mathcal{G}(I(P_n^k))| = nk - \frac{k(k+1)}{2}$.*

Remark 2.2. If $n \geq 2k+1$, then $|\mathcal{G}(I(C_n^k))| = nk$.

Let G be a graph and $i \in [n]$, then $N_G(x_i) := \{x_j : x_ix_j \in \mathcal{G}\big(I(G)\big)\}$, where $j \in [n]\backslash\{i\}$. For $k \geq 2$, $0 \leq i \leq k-1$ and $n \geq 2k+2$, let A_{n-k-1},

A_{n-k+i}, B_{n-k+i} and D_{n-k+i} denote the monomial prime ideals of S, such that $A_{n-k-1} = (0)$, $A_{n-k+i} := (x_{n-k}, x_{n-k+1}, \dots, x_{n-k+i})$,

$$\begin{aligned} B_{n-k+i} &:= \big(x_j : x_j \in N_{P_n^k}(x_{n-k+i})\big) \\ &= (x_{n-2k+i}, x_{n-2k+i+1}, \dots, x_{n-k+i-1}, x_{n-k+i+1}, \dots, x_n), \end{aligned}$$

and $D_{n-k+i} := \big(x_j : x_j \in N_{C_n^k}(x_{n-k+i})\big)$. Thus if $i = 0$, then

$$D_{n-k+i} = (x_{n-2k}, x_{n-2k+1}, \dots, x_{n-k-1}, x_{n-k+1}, \dots, x_n)$$

and if $1 \le i \le k-1$, then

$$D_{n-k+i} = (x_{n-2k+i}, x_{n-2k+i+1}, \dots, x_{n-k+i-1}, x_{n-k+i+1}, \dots, x_n, x_1, \dots, x_i).$$

These monomial prime ideals and the following function play important role in the proof of our main theorems. For $k \ge 2$ and $2k+2 \le n \le 3k+1$, we define a function

$$f : \{n-k, n-k+1, \dots, n-k+i, \dots, n-1\} \longrightarrow \mathbb{Z}^+ \cup \{0\}, \text{ by}$$

$$f(n-k+i) = \begin{cases} k, & \text{if } n-2k-1+i \ge k+1; \\ n-2k-2+i, & \text{if } 2 \le n-2k-1+i < k+1. \end{cases}$$

In the following we recall some known results that we refer several times in this paper.

Lemma 2.3 (Depth Lemma). *If $0 \longrightarrow U \longrightarrow M \longrightarrow N \longrightarrow 0$ is a short exact sequence of modules over a local ring S, or a Noetherian graded ring with S_0 local, then*

1. $\operatorname{depth} M \ge \min\{\operatorname{depth} N, \operatorname{depth} U\}$.
2. $\operatorname{depth} U \ge \min\{\operatorname{depth} M, \operatorname{depth} N + 1\}$.
3. $\operatorname{depth} N \ge \min\{\operatorname{depth} U - 1, \operatorname{depth} M\}$.

Lemma 2.4 ([23, Lemma 2.2]). *Let $0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$ be a short exact sequence of $\mathbb{Z}^n$-graded S-modules. Then*

$$\operatorname{sdepth}(V) \ge \min\{\operatorname{sdepth}(U), \operatorname{sdepth}(W)\}.$$

Lemma 2.5 ([13, Lemma 3.3]). *Let $I \subset S$ be a squarefree monomial ideal with $\operatorname{supp}(I) = [n]$ and $v := x_{i_1} x_{i_2} \cdots x_{i_q} \in S/I$ such that $x_j v \in I$ for all $j \in [n] \setminus \operatorname{supp}(v)$. Then $\operatorname{sdepth}(S/I) \le q$.*

The above Lemma can also be seen as an immediate consequence of the result of J. Apel [1, Sec.3].

3 Depth and Stanley of cyclic modules associated to the edge ideals of the powers of a path

We start this section with some results. These results are essential for computations of depth and Stanley depth of $S/I(P_n^k)$.

Lemma 3.1. *Let $a \geq 2$ be an integer, $\{E_i : 1 \leq i \leq a\}$ and $\{G_i : 0 \leq i \leq a\}$ be some families of $\mathbb{Z}^n$-graded S-modules such that we have the following short exact sequences:*

$$0 \longrightarrow E_1 \longrightarrow G_0 \longrightarrow G_1 \longrightarrow 0 \qquad (1)$$

$$0 \longrightarrow E_2 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow 0 \qquad (2)$$

$$\vdots$$

$$0 \longrightarrow E_{a-1} \longrightarrow G_{a-2} \longrightarrow G_{a-1} \longrightarrow 0 \qquad (a-1)$$

$$0 \longrightarrow E_a \longrightarrow G_{a-1} \longrightarrow G_a \longrightarrow 0 \qquad (a)$$

and $\operatorname{depth}(G_a) \geq \operatorname{depth}(E_a)$, $\operatorname{depth}(E_i) \geq \operatorname{depth}(E_{i-1})$ *for all* $2 \leq i \leq a$. *Then* $\operatorname{depth}(G_0) = \operatorname{depth}(E_1)$.

Proof. By assumption, we have $\operatorname{depth}(G_a) \geq \operatorname{depth}(E_a)$, applying Depth Lemma on the exact sequence (a) we get $\operatorname{depth}(G_{a-1}) = \operatorname{depth}(E_a)$. We also have by assumption

$$\operatorname{depth}(G_{a-1}) = \operatorname{depth}(E_a) \geq \operatorname{depth}(E_{a-1}).$$

By applying Depth Lemma on the exact sequence $(a-1)$ we have $\operatorname{depth}(G_{a-2}) = \operatorname{depth}(E_{a-1})$. We repeat the same steps on all exact sequences one by one from bottom to top and we get $\operatorname{depth}(G_{i-1}) = \operatorname{depth}(E_i)$ for all i. Thus if $i = 1$ then we have $\operatorname{depth}(G_0) = \operatorname{depth}(E_1)$. $\square$

Lemma 3.2. *Let $k \geq 2$ and $n \geq 2k+2$. Then*

$$S/(I(P_n^k), A_{n-1}) \cong S_{n-k-1}/I(P_{n-k-1}^k)[x_n].$$

Proof. Since

$$\mathcal{G}(I(P_n^k)) = \cup_{i=1}^{n-k}\{x_ix_{i+1}, x_ix_{i+2}, \dots, x_ix_{i+k}\} \cup \cup_{i=n-k+1}^{n-1}\{x_ix_{i+1}, x_ix_{i+2}, \dots, x_ix_n\},$$

so we have

$$\begin{aligned}&I(P_n^k)+A_{n-1}=A_{n-1}+\\&\Big[\sum_{i=1}^{n-2k-1}(x_ix_{i+1},x_ix_{i+2},\dots,x_ix_{i+k})+\sum_{i=n-2k}^{n-k}(x_ix_{i+1},x_ix_{i+2},\dots,x_ix_{i+k})+\\&\sum_{i=n-k+1}^{n-1}(x_ix_{i+1},x_ix_{i+2},\dots,x_ix_n)\Big]=\sum_{i=1}^{n-2k-1}\big(x_ix_{i+1},x_ix_{i+2},\dots,x_ix_{i+k}\big)+\\&\qquad\sum_{i=n-2k}^{n-k-2}(x_ix_{i+1},x_ix_{i+2},\dots,x_ix_{n-k-1})+A_{n-1}=I(P_{n-k-1}^k)+A_{n-1}.\end{aligned}$$

Thus the required result follows. □

Lemma 3.3. *Let $k\geq 2$, $0\leq i\leq k-1$ and $n\geq 3k+2$. Then*

$$S/(I(P_n^k):x_{n-k+i})\cong S_{n-2k-1+i}/I(P_{n-2k-1+i}^k)[x_{n-k+i}].$$

Proof. It is enough to prove that $(I(P_n^k):x_{n-k+i})=(I(P_{n-2k-1+i}^k),B_{n-k+i})$. Clearly

$$I(P_{n-2k-1+i}^k)\subset I(P_n^k)\subset(I(P_n^k):x_{n-k+i}).$$

Let $u\in B_{n-k+i}$, then by definition of $I(P_n^k)$, $ux_{n-k+i}\in I(P_n^k)$ that is $u\in(I(P_n^k):x_{n-k+i})$. Thus $B_{n-k+i}\subset(I(P_n^k):x_{n-k+i})$ and we have $\big(I(P_{n-2k-1+i}^k),B_{n-k+i}\big)\subset(I(P_n^k):x_{n-k+i})$. Now for the other inclusion, let w be a monomial generator of $(I(P_n^k):x_{n-k+i})$, then $w=\frac{v}{gcd(v,x_{n-k+i})}$, where $v\in\mathcal{G}(I(P_n^k))$. If $\mathrm{supp}(v)\cap\mathcal{G}(B_{n-k+i})\neq\emptyset$, then we have $w\in\mathcal{G}(B_{n-k+i})$ and if $\mathrm{supp}(v)\cap\mathcal{G}(B_{n-k+i})=\emptyset$, then $w\in\mathcal{G}(I(P_n^k))\cap K[x_1,x_2,\dots,x_{n-2k-1+i}]=\mathcal{G}(I(P_{n-2k-1+i}^k))$. □

Lemma 3.4. *Let $n\geq 3k+2$ and $0\leq i\leq k-1$, then we have*

$$S/((I(P_n^k),A_{n-k+(i-1)}):x_{n-k+i})\cong S_{n-2k-1+i}/I(P_{n-2k-1+i}^k)[x_{n-k+i}].$$

Proof. As $\big((I(P_n^k),A_{n-k+(i-1)}):x_{n-k+i}\big)=\big((I(P_n^k):x_{n-k+i}),A_{n-k+(i-1)}\big)$. Now using the proof of Lemma 3.3 we obtain

$$\begin{aligned}&\big((I(P_n^k):x_{n-k+i}),A_{n-k+(i-1)}\big)=\\&\qquad\big(I(P_{n-2k-1+i}^k),B_{n-k+i},A_{n-k+(i-1)}\big)=\big(I(P_{n-2k-1+i}^k),B_{n-k+i}\big),\end{aligned}$$

as $A_{n-k+(i-1)}\subset B_{n-k+i}$. Thus the required result follows by Lemma 3.3. □

Remark 3.5. Let $m \geq 2$ and $I(P_m^{m-1}) \subset S_m = K[x_1, x_2, \ldots, x_m]$ be the edge ideal of the $(m-1)^{th}$ power of path P_m. Then $I(P_m^{m-1})$ is a squarefree Veronese ideal of degree 2 in variables $x_1, x_2, \ldots, x_m$. Thus by [10, Corollary 10.3.7] and Theorem 3.9

$$\operatorname{depth}(S_m/I(P_m^{m-1})) = \operatorname{sdepth}(S_m/I(P_m^{m-1})) = 1.$$

Remark 3.6. Let $k \geq 2$ and $2k+2 \leq n \leq 3k+1$, then it is easy to see that

(1) If $n = 2k+2$, then

$$S/(I(P_n^k) : x_{n-k}) = S/(x_2, \ldots, x_{n-k-1}, x_{n-k+1}, \ldots, x_n) \cong K[x_1, x_{n-k}].$$

(2) If $0 \leq i \leq k-1$ and $n > 2k+2$, then

$$\begin{aligned} S/(I(P_n^k) : x_{n-k+i}) &= S/((I(P_n^k), A_{n-k+(i-1)}) : x_{n-k+i}) \\ &\cong S_{n-2k-1+i}/I(P_{n-2k-1+i}^{f(n-k+i)})[x_{n-k+i}] \end{aligned}$$
$$= \begin{cases} S_{n-2k-1+i}/I(P_{n-2k-1+i}^{k})[x_{n-k+i}], & \text{if } n-2k-1+i \geq k+1; \\ S_{n-2k-1+i}/I(P_{n-2k-1+i}^{n-2k-2+i})[x_{n-k+i}], & \text{otherwise.} \end{cases}$$

We recall a lemma from [11] which is heavily used in this paper.

Lemma 3.7 ([11, Lemma 3.6])**.** *Let $J \subset I$ be monomial ideals of S, and let $T = S[x_{n+1}]$ be the polynomial ring over S in the variable x_{n+1}. Then* $\operatorname{depth}(IT/JT) = \operatorname{depth}(I/J) + 1$ *and* $\operatorname{sdepth}(IT/JT) = \operatorname{sdepth}(I/J) + 1$.

Theorem 3.8. *Let $n \geq 2$. Then* $\operatorname{depth}(S/I(P_n^k)) = \lceil \frac{n}{2k+1} \rceil$.

Proof. (a) If $n \leq k+1$, then $I(P_n^k)$ is a squarefree Veronese ideal thus by Remark 3.5, $\operatorname{depth}(S/I(P_n^k)) = 1 = \lceil \frac{n}{2k+1} \rceil$.

(b) For $n \geq k+2$, we consider the following cases:

(1) If $k = 1$, then by [18, Lemma 2.8] we have $\operatorname{depth}(S/I(P_n^1)) = \lceil \frac{n}{3} \rceil = \lceil \frac{n}{2k+1} \rceil$.

(2) If $k \geq 2$ and $k+2 \leq n \leq 2k+1$, then we get $\operatorname{depth}(S/I(P_n^k)) \geq 1$ as $\mathfrak{m} \notin \operatorname{Ass}(S/I(P_n^k))$. Since $x_{k+1} \notin I(P_n^k)$ and $x_s x_{k+1} \in \mathcal{G}(I(P_n^k))$ for all $s \in \{1, \ldots k, k+2, \ldots, n\}$, therefore we have $(I(P_n^k) : x_{k+1}) = (x_1, \ldots x_k, x_{k+2}, \ldots, x_n)$. By [23, Corollary 1.3], we have

$$\begin{aligned} \operatorname{depth}(S/I(P_n^k)) &\leq \operatorname{depth}(S/(I(P_n^k) : x_{k+1})) \\ &= \operatorname{depth}(S/(x_1, \ldots x_k, x_{k+2}, \ldots, x_n)) = 1. \end{aligned}$$

Thus $\operatorname{depth}(S/I(P_n^k)) = 1 = \lceil \frac{n}{2k+1} \rceil$.

(3) For $k \geq 2$, $2k+2 \leq n \leq 3k+1$ and $0 \leq i \leq k-1$, consider the family of short exact sequences

$$0 \longrightarrow S/((I(P_n^k), A_{n-k+(i-1)}) : x_{n-k+i}) \xrightarrow{\cdot x_{n-k+i}} S/(I(P_n^k), A_{n-k+(i-1)}) \longrightarrow S/(I(P_n^k), A_{n-k+i}) \longrightarrow 0$$

By Lemma 3.2, $S/(I(P_n^k), A_{n-1}) \cong S_{n-k-1}/I(P_{n-k-1}^k)[x_n]$. Since we are considering the case $2k+2 \leq n \leq 3k+1$ which implies that $k+1 \leq n-k-1 \leq 2k$. If $n-k-1 = k+1$ then $S_{n-k-1}/I(P_{n-k-1}^k) = S_{k+1}/I(P_{k+1}^k)$, by Remark 3.5 and Lemma 3.7 we have $\operatorname{depth}\big(S/(I(P_n^k), A_{n-1})\big) = 2$. If $k+1 < n-k-1 \leq 2k$, then by case(b)(2) $\operatorname{depth}(S_{n-k-1}/I(P_{n-k-1}^k)) = 1$. Thus by Lemma 3.7 we have $\operatorname{depth}\big(S/(I(P_n^k), A_{n-1})\big) = 2$. Now we show that $\operatorname{depth}\big(S/(I(P_n^k) : x_{n-k})\big) = 2$. For this we consider two cases: If $n = 2k+2$, then by Remark 3.6

$$S/(I(P_n^k) : x_{n-k}) = S/(x_2, x_3, \dots, x_{n-k-1}, x_{n-k+1}, \dots, x_n) \cong K[x_1, x_{n-k}],$$

and thus $\operatorname{depth}\big(S/(I(P_n^k) : x_{n-k})\big) = 2$. If $n > 2k+2$, by Remark 3.6 we have

$$S/(I(P_n^k) : x_{n-k}) \cong S_{n-2k-1}/I(P_{n-2k-1}^{n-2k-2})[x_{n-k}],$$

where $2 \leq n-2k-1 \leq k$. Thus by Remark 3.5 and Lemma 3.7 we get $\operatorname{depth}\big(S/(I(P_n^k) : x_{n-k})\big) = 2$. Now for $1 \leq i \leq k-1$, by Remark 3.6 we obtain

$$S/((I(P_n^k), A_{n-k+(i-1)}) : x_{n-k+i}) = S/(I(P_n^k) : x_{n-k+i}) \cong S_{n-2k-1+i}/I(P_{n-2k-1+i}^{f(n-k+i)})[x_{n-k+i}].$$

Let $T := S_{n-2k-1+i}/I(P_{n-2k-1+i}^{f(n-k+i)})[x_{n-k+i}]$. We consider the following cases:

(i) If $k+1 = n-2k-1+i$, then $T = S_{k+1}/I(P_{k+1}^k)[x_{n-k+i}]$, thus by case(a) and Lemma 3.7 we have $\operatorname{depth}(T) = 2$.

(ii) For $k+1 < n-2k-1+i$, $T = S_{n-2k-1+i}/I(P_{n-2k-1+i}^k)[x_{n-k+i}]$. Since $k+2 \leq n-2k-1+i \leq 2k-1$, thus by case(b)(2) and Lemma 3.7 we have $\operatorname{depth}(T) = 2$.

(iii) If $2 \leq n-2k-1+i < k+1$, then $T = S_{n-2k-1+i}/I(P_{n-2k-1+i}^{n-2k-2+i})[x_{n-k+i}]$, by Remark 3.5 and Lemma 3.7 we have $\operatorname{depth}(T) = 2$.

Thus by Lemma 3.1 we have $\operatorname{depth}(S/I(P_n^k)) = 2$.

(4) For $k \geq 2$, $n \geq 3k+2$ and $0 \leq i \leq k-1$, consider the family of short exact sequences

$$0 \longrightarrow S/((I(P_n^k), A_{n-k+(i-1)}) : x_{n-k+i}) \xrightarrow{\cdot x_{n-k+i}} S/(I(P_n^k), A_{n-k+(i-1)}) \longrightarrow S/(I(P_n^k), A_{n-k+i}) \longrightarrow 0$$

By Lemma 3.2, $S/(I(P_n^k), A_{n-1})) \cong S_{n-k-1}/I(P_{n-k-1}^k)[x_n]$. Thus by induction on n and Lemma 3.7 we have $\operatorname{depth}(S/(I(P_n^k), A_{n-1})) = \lceil \frac{n-k-1}{2k+1} \rceil + 1$. By Lemma 3.4 we have

$$S/((I(P_n^k), A_{n-k+(i-1)}) : x_{n-k+i}) \cong S_{n-2k-1+i}/I(P_{n-2k-1+i}^k)[x_{n-k+i}].$$

Thus by induction on n and Lemma 3.7 we have

$$\operatorname{depth}(S/((I(P_n^k), A_{n-k+(i-1)}) : x_{n-k+i})) = \lceil \frac{n-2k-1+i}{2k+1} \rceil + 1.$$

Here we can see that

$$\operatorname{depth}(S/(I(P_n^k), A_{n-1})) = \lceil \tfrac{n-k-1}{2k+1} \rceil + 1 \geq \lceil \frac{n-k-2}{2k+1} \rceil + 1 = \operatorname{depth}(S/(I(P_n^k), A_{n-2}) : x_{n-1})),$$

and for all $1 \leq i \leq k-1$,

$$\operatorname{depth}(S/((I(P_n^k), A_{n-k+(i-1)}) : x_{n-k+i})) = \lceil \tfrac{n-2k-1+i}{2k+1} \rceil + 1 \geq \lceil \frac{n-2k-2+i}{2k+1} \rceil + 1 = \operatorname{depth}(S/((I(P_n^k), A_{n-k+(i-2)}) : x_{n-k+(i-1)})).$$

Thus by Lemma 3.1 we have $\operatorname{depth}(S/I(P_n^k)) = \lceil \frac{n-2k-1}{2k+1} \rceil + 1 = \lceil \frac{n}{2k+1} \rceil$.

□

Let $d \in [n]$ and $I_{n,d} := (u \in S \text{ square free monomial} : \deg(u) = d)$. Then $I_{n,d}$ is called squarefree Veronese ideal of degree d in the variables $x_1, x_2, \ldots, x_n$. Cimpoeas proved the following theorems:

Theorem 3.9 ([5, Theorem 1.1]). *(1)* $\operatorname{sdepth}(S/I_{n,d}) = d-1$.

(2) $d \leq \operatorname{sdepth}(I_{n,d}) \leq \frac{n-d}{d+1} + d$.

Theorem 3.10 ([7, Theorem 1.4]). *Let M be a $\mathbb{Z}^n$-graded S-module. If* $\operatorname{sdepth}(M) = 0$, *then* $\operatorname{depth}(M) = 0$. *Conversely, if* $\operatorname{depth}(M) = 0$ *and* $\dim_K(M_a) = 1$ *for any $a \in \mathbb{Z}^n$, then* $\operatorname{sdepth}(M) = 0$.

Lemma 3.11 ([25, Lemma 4]). *Let $n \geq 2$, then* $\operatorname{sdepth}(S/I(P_n^1)) = \lceil \frac{n}{3} \rceil$.

Example 3.12. *Let $n \geq 2$, and $n \leq 2k+1$, then* $\operatorname{sdepth}(S/I(P_n^k)) = 1$.

Proof. If $n \leq k+1$, then by Theorem 3.9 $\operatorname{sdepth}(S/I(P_n^k)) = 1$. Now if $k+2 \leq n \leq 2k+1$, then $\operatorname{depth}(S/I(P_n^k)) \geq 1$ as $\mathfrak{m} \notin \operatorname{Ass}(S/I(P_n^k))$, thus by Theorem 3.10 $\operatorname{sdepth}(S/I(P_n^k)) \geq 1$. Since $x_{k+1} \notin I(P_n^k)$ and $x_i x_{k+1} \in \mathcal{G}(I(P_n^k))$ for all $i \in \{1, \dots k, k+2, \dots, n\}$, therefore $(I(P_n^k) : x_{k+1}) = (x_1, \dots x_k, x_{k+2}, \dots, x_n)$. Thus by [4, Proposition 2.7] $\operatorname{sdepth}(S/I(P_n^k)) \leq \operatorname{sdepth}(S/(I(P_n^k) : x_{k+1})) = \operatorname{sdepth}(S/(x_1, \dots x_k, x_{k+2}, \dots, x_n)) = 1$. □

Proposition 3.13. *Let $k \geq 2$ and $n \geq 2k+2$. Then*

$$\operatorname{sdepth}(S/I(P_n^k)) \geq \lceil \frac{n}{2k+1} \rceil.$$

Proof. (1) If $2k+2 \leq n \leq 3k+1$, then by applying Lemma 2.4 on the exact sequences in case(b)(3) of Theorem 3.8 we get $\operatorname{sdepth}(S/I(P_n^k)) \geq 2 = \lceil \frac{n}{2k+1} \rceil$.

(2) If $n \geq 3k+2$, then the proof is similar to Theorem 3.8. We apply Lemma 2.4 on the exact sequences in case(b)(4) of Theorem 3.8 and obtain

$$\operatorname{sdepth}(S/I(P_n^k)) \geq \min \Big\{ \operatorname{sdepth}(S/(I(P_n^k), A_{n-1})), \\ \min_{i=0}^{k-1} \{\operatorname{sdepth}(S/((I(P_n^k), A_{n-k+(i-1)}) : x_{n-k+i}))\} \Big\} \geq \lceil \frac{n}{2k+1} \rceil.$$

□

Theorem 3.14. *Let $n \geq 2$, then* $\operatorname{sdepth}(S/I(P_n^k)) = \lceil \frac{n}{2k+1} \rceil$.

Proof. If $k = 1$, then the result follows by Lemma 3.11. Let $k \geq 2$. If $n \leq 2k+1$, then by Example 3.12 we have the required result. If $n \geq 2k+2$, then by Proposition 3.13 we have

$$\operatorname{sdepth}(S/I(P_n^k)) \geq \lceil \frac{n}{2k+1} \rceil.$$

We need to prove that $\operatorname{sdepth}(S/I(P_n^k)) \leq \lceil \frac{n}{2k+1} \rceil$, for this we consider the following three cases:

(1) If $n = (2k+1)l$, where $l \geq 1$. We see that

$$v = x_{k+1}x_{3k+2}x_{5k+3}\cdots x_{(2k+1)l-k} \in S\backslash I(P_n^k),$$

but $x_{t_1}v \in I(P_n^k)$ for all $t_1 \in [n]\backslash \operatorname{supp}(v)$, thus by Lemma 2.5,

$$\operatorname{sdepth}(S/I(P_n^k)) \leq l = \lceil \frac{n}{2k+1} \rceil.$$

(2) If $n = (2k+1)l + r$, where $r \in \{1, 2, 3, \ldots, k+1\}$ and $l \geq 1$, then we have

$$v = x_{k+1}x_{3k+2}x_{5k+3}\cdots x_{(2k+1)l-k}x_{(2k+1)l+r} \in S\backslash I(P_n^k),$$

and $x_{t_2}v \in I(P_n^k)$ for all $t_2 \in [n]\backslash \operatorname{supp}(v)$, so by Lemma 2.5,

$$\operatorname{sdepth}(S/I(P_n^k)) \leq l + 1 = \lceil \frac{n}{2k+1} \rceil.$$

(3) If $n = (2k+1)l + s$, where $s \in \{k+2, k+3, \ldots, 2k\}$ and $l \geq 1$, since

$$v = x_{k+1}x_{3k+2}x_{5k+3}\cdots x_{(2k+1)l+k+1} \in S\backslash I(P_n^k),$$

but $x_{t_3}v \in I(P_n^k)$ for all $t_3 \in [n]\backslash \operatorname{supp}(v)$, by Lemma 2.5, we get

$$\operatorname{sdepth}(S/I(P_n^k)) \leq l + 1 = \lceil \frac{n}{2k+1} \rceil.$$

□

4 Depth and Stanley depth of cyclic modules associated to the edge ideals of the powers of a cycle

In this section, we compute bounds for depth and Stanley depth of cyclic modules associated to the edge ideals of powers of a cycle. In order to complete the main task of this section we prove the following three lemmas.

Lemma 4.1. *Let $k \geq 2$ and $n \geq 3k+2$, then $S/(I(C_n^k), A_{n-1}) \cong S_{n-k}/I(P_{n-k}^k)$.*

Proof. Since $\mathcal{G}(I(C_n^k)) = \mathcal{G}(I(P_n^k)) \cup \cup_{l=1}^{k-1}\{x_l x_{l+n-k}, x_l x_{l+n-k+1}, \ldots, x_l x_{n-1}\} \cup \{x_1x_n, x_2x_n, \ldots, x_kx_n\}$, we have

$$I(C_n^k) + A_{n-1} =$$

$$I(P_n^k) + \sum_{l=1}^{k-1}(x_l x_{l+n-k}, x_l x_{l+n-k+1}, \ldots, x_l x_{n-1}) + (x_1x_n, x_2x_n, \ldots, x_kx_n) + A_{n-1}.$$

Thus by the proof of Lemma 3.2, we obtain $I(P_n^k)+A_{n-1} = I(P_{n-k-1}^k)+A_{n-1}$. As

$$\sum_{l=1}^{k-1}(x_lx_{l+n-k}, x_lx_{l+n-k+1}, \dots, x_lx_{n-1}) + A_{n-1} = A_{n-1}.$$

Therefore $S/(I(C_n^k), A_{n-1}) = S/(I(P_{n-k-1}^k), A_{n-1}, (x_1x_n, x_2x_n, \dots, x_kx_n))$
$\cong K[x_1, x_2, \dots, x_{n-k-1}, x_n]/\big(I(P_{n-k-1}^k), (x_1x_n, x_2x_n, \dots, x_kx_n)\big)$.

After renumbering the variables, we have

$$K[x_1, \dots, x_{n-k-1}, x_n]/\big(I(P_{n-k-1}^k), (x_1x_n, x_2x_n, \dots, x_kx_n)\big) \cong S_{n-k}/I(P_{n-k}^k).$$

□

Lemma 4.2. *Let $k \geq 2$ and $n \geq 3k+2$ and $0 \leq i \leq k-1$, then*

$$S/(I(C_n^k) : x_{n-k+i}) \cong S_{n-2k-1}/I(P_{n-2k-1}^k)[x_{n-k+i}].$$

Proof. Let w be a monomial generator of $(I(C_n^k) : x_{n-k+i})$. Then $w = \frac{v}{gcd(v, x_{n-k+i})}$, where $v \in \mathcal{G}(I(C_n^k))$. If $\mathrm{supp}(v) \cap \mathcal{G}(D_{n-k+i}) \neq \emptyset$, then we have $w \in \mathcal{G}(D_{n-k+i})$ and if $\mathrm{supp}(v) \cap \mathcal{G}(D_{n-k+i}) = \emptyset$ then $w \in E := \mathcal{G}(I(C_n^k)) \cap K[x_{i+1}, x_{i+2}, \dots, x_{n-2k-1+i}]$. So we obtain $(I(C_n^k) : x_{n-k+i}) \subset E + D_{n-k+i}$. The other inclusion being trivial we get $(I(C_n^k) : x_{n-k+i}) = E + D_{n-k+i}$, which further implies that $S/(I(C_n^k) : x_{n-k+i}) = S/(E+D_{n-k+i})$. After renumbering the variables, we have

$$S/(I(C_n^k) : x_{n-k+i}) = S/(E, D_{n-k+i}) \cong S_{n-2k-1}/I(P_{n-2k-1}^k)[x_{n-k+i}].$$

□

Lemma 4.3. *Let $k \geq 2$, $n \geq 3k+2$ and $0 \leq i \leq k-1$. Then*

$$S/\big((I(C_n^k), A_{n-k+(i-1)}) : x_{n-k+i}\big) \cong S_{n-2k-1}/I(P_{n-2k-1}^k)[x_{n-k+i}].$$

Proof. As $\big((I(C_n^k), A_{n-k+(i-1)}) : x_{n-k+i}\big) = \big((I(C_n^k) : x_{n-k+i}), A_{n-k+(i-1)}\big)$. By using the same arguments as in the proof of Lemma 4.2 we have

$$\big((I(C_n^k) : x_{n-k+i}), A_{n-k+(i-1)}\big) = \big(E, D_{n-k+i}, A_{n-k+(i-1)}\big) = \big(E, D_{n-k+i}\big)$$

as $A_{n-k+(i-1)} \subset D_{n-k+i}$. Thus the required result follows by Lemma 4.2. □

Corollary 4.4 ([10, Corollary 10.3.7])**.** *Let $2 \leq d < n$. Then*

$$\mathrm{depth}(S/I_{n,d}^t) = \max\{0, n - t(n-d) - 1\}.$$

Theorem 4.5. *Let $n \geq 3$, then*

$$\begin{aligned} \operatorname{depth}(S/I(C_n^k)) &= 1, \quad \text{if } n \leq 2k+1; \\ \operatorname{depth}(S/I(C_n^k)) &\geq \lceil \frac{n-k}{2k+1} \rceil, \quad \text{if } n \geq 2k+2. \end{aligned}$$

Proof. (a) If $n \leq 2k+1$, then $I(C_n^k)$ is a squarefree Veronese ideal of degree 2. Thus by Corollary 4.4, $\operatorname{depth}(S/I(C_n^k)) = 1$.

(b) For $n \geq 2k+2$, we consider the following cases:

(1) If $k = 1$, then by [6, Proposition 1.3] $\operatorname{depth}(S/I(C_n^1)) = \lceil \frac{n-1}{3} \rceil$.

(2) If $k \geq 2$ and $2k+2 \leq n \leq 3k+1$, then we have $\operatorname{depth}(S/I(C_n^k)) \geq 1 = \lceil \frac{n-k}{2k+1} \rceil$ as $\mathfrak{m} \notin \operatorname{Ass}(S/I(C_n^k))$.

(3) For $k \geq 2$, $n \geq 3k+2$ and $0 \leq i \leq k-1$, consider the family of short exact sequences

$$0 \longrightarrow S/((I(C_n^k), A_{n-k+(i-1)}) : x_{n-k+i}) \xrightarrow{\cdot x_{n-k+i}} S/(I(C_n^k), A_{n-k+(i-1)}) \longrightarrow S/(I(C_n^k), A_{n-k+i}) \longrightarrow 0$$

By Lemma 4.1 we have $S/(I(C_n^k), A_{n-1}) \cong S_{n-k}/I(P_{n-k}^k)$. Now by Lemma 4.3, we get

$$S/((I(C_n^k), A_{n-k+(i-1)}) : x_{n-k+i}) \cong S_{n-2k-1}/I(P_{n-2k-1}^k)[x_{n-k+i}].$$

By Theorem 3.8 and Lemma 3.7, we obtain

$$\operatorname{depth}(S/((I(C_n^k), A_{n-k+(i-1)}) : x_{n-k+i})) = \lceil \frac{n-2k-1}{2k+1} \rceil + 1 = \lceil \frac{n}{2k+1} \rceil.$$

Again by Theorem 3.8, we have $\operatorname{depth}(S/(I(C_n^k), A_{n-1})) = \lceil \frac{n-k}{2k+1} \rceil$. Thus by applying Lemma 2.3(1) on the family of short exact sequences we get $\operatorname{depth}(S/I(C_n^k)) \geq \lceil \frac{n-k}{2k+1} \rceil$.

□

Corollary 4.6. *Let $n \geq 3$, If $n \geq 2k+2$, then*

$$\operatorname{depth}(S/I(C_n^k)) = \lceil \frac{n}{2k+1} \rceil, \quad \text{if } n \equiv 0, k+1, \dots, 2k \,(\operatorname{mod}(2k+1));$$

$$\lceil \frac{n}{2k+1} \rceil - 1 \leq \operatorname{depth}(S/I(C_n^k)) \leq \lceil \frac{n}{2k+1} \rceil, \quad \text{if } n \equiv 1, \dots, k \,(\operatorname{mod}(2k+1)).$$

Proof. By Theorem 4.5, it is enough to prove that $\operatorname{depth}(S/I(C_n^k)) \leq \lceil \frac{n}{2k+1} \rceil$, for $k \geq 2$ and $n \geq 2k+2$. Since $x_{n-k} \notin I(C_n^k)$, thus by [23, Corollary 1.3] we have $\operatorname{depth}(S/I(C_n^k)) \leq \operatorname{depth}(S/(I(C_n^k) : x_{n-k}))$. Now we consider two cases:

(1) Let $2k+2 \leq n \leq 3k+1$, then $S/(I(C_n^k) : x_{n-k}) = S/(I(P_n^k) : x_{n-k})$ so by the proof of Theorem 3.8 we have $\operatorname{depth}(S/(I(P_n^k) : x_{n-k})) = 2 = \lceil \frac{n}{2k+1} \rceil$. Therefore

$$\operatorname{depth}(S/I(C_n^k)) \leq \operatorname{depth}(S/(I(C_n^k) : x_{n-k})) = 2 = \lceil \frac{n}{2k+1} \rceil.$$

(2) Let $n \geq 3k+2$, then by Lemma 4.2,

$$S/(I(C_n^k) : x_{n-k}) \cong S_{n-2k-1}/I(P_{n-2k-1}^k)[x_{n-k}].$$

By Lemma 3.7 and Theorem 3.8, $\operatorname{depth}(S_{n-2k-1}/I(P_{n-2k-1}^k)[x_{n-k}]) = \lceil \frac{n}{2k+1} \rceil$. Thus $\operatorname{depth}(S/I(C_n^k)) \leq \operatorname{depth}(S/(I(C_n^k) : x_{n-k})) = \lceil \frac{n}{2k+1} \rceil$.

□

Theorem 4.7. *Let $n \geq 3$, then*

$$\begin{aligned} \operatorname{sdepth}(S/I(C_n^k)) &= 1, && \textit{if } n \leq 2k+1; \\ \operatorname{sdepth}(S/I(C_n^k)) &\geq \lceil \frac{n-k}{2k+1} \rceil, && \textit{if } n \geq 2k+2. \end{aligned}$$

Proof. (a) If $n \leq 2k+1$, then $\operatorname{sdepth}(S/I(C_n^k)) = 1$ by Theorem 3.9.

(b) For $n \geq 2k+2$, consider the following cases:

(1) If $k = 1$, then by [6, Proposition 1.8] $\operatorname{sdepth}(S/I(C_n^1)) \geq \lceil \frac{n-1}{3} \rceil$.

(2) If $k \geq 2$ and $2k+2 \leq n \leq 3k+1$, then $\operatorname{depth}(S/I(C_n^k)) \geq 1$ as $\mathfrak{m} \notin \operatorname{Ass}(S/I(C_n^k))$, thus by Theorem 3.10, $\operatorname{sdepth}(S/I(C_n^k)) \geq 1 = \lceil \frac{n-k}{2k+1} \rceil$.

(3) For $k \geq 2$, $n \geq 3k+2$ and $0 \leq i \leq k-1$, consider the family of short exact sequences

$$0 \longrightarrow S/((I(C_n^k), A_{n-k+(i-1)}) : x_{n-k+i}) \xrightarrow{\cdot x_{n-k+i}} S/(I(C_n^k), A_{n-k+(i-1)}) \longrightarrow S/(I(C_n^k), A_{n-k+i}) \longrightarrow 0.$$

By Lemma 4.1 we have $S/(I(C_n^k), A_{n-1})) \cong S_{n-k}/I(P_{n-k}^k)$. Now by Lemma 4.3, we get

$$S/((I(C_n^k), A_{n-k+(i-1)}) : x_{n-k+i}) \cong S_{n-2k-1}/I(P_{n-2k-1}^k)[x_{n-k+i}].$$

By Theorem 3.14 and Lemma 3.7, we obtain

$$\operatorname{sdepth}(S/((I(C_n^k), A_{n-k+(i-1)}) : x_{n-k+i})) = \lceil \frac{n-2k-1}{2k+1} \rceil + 1 = \lceil \frac{n}{2k+1} \rceil.$$

Again by Theorem 3.14, we have $\mathrm{sdepth}(S/(I(C_n^k), A_{n-1})) = \lceil \frac{n-k}{2k+1} \rceil$. By applying Lemma 2.4 on the above family of short exact sequences we get $\mathrm{sdepth}(S/I(C_n^k)) \geq \lceil \frac{n-k}{2k+1} \rceil$.

□

Corollary 4.8. *Let $n \geq 3$, if $n \geq 2k+2$, then*

$$\mathrm{sdepth}(S/I(C_n^k)) = \lceil \frac{n}{2k+1} \rceil, \qquad \text{if } n \equiv 0, k+1, \ldots, 2k \,(\mathrm{mod}(2k+1));$$

$$\lceil \frac{n}{2k+1} \rceil - 1 \leq \mathrm{sdepth}(S/I(C_n^k)) \leq \lceil \frac{n}{2k+1} \rceil, \text{ if } n \equiv 1, \ldots, k \,(\mathrm{mod}(2k+1)).$$

Proof. When $k = 1$, then by [6, Theorem 1.9], $\mathrm{sdepth}(S/I(C_n^k)) \leq \lceil \frac{n}{3} \rceil$. By Theorem 4.7 it is enough to prove that $\mathrm{sdepth}(S/I(C_n^k)) \leq \lceil \frac{n}{2k+1} \rceil$ for $k \geq 2$ and $n \geq 2k+2$. Since $x_{n-k} \notin I(C_n^k)$, thus by [4, Proposition 2.7] we have

$$\mathrm{sdepth}(S/I(C_n^k)) \leq \mathrm{sdepth}(S/(I(C_n^k) : x_{n-k})).$$

Now we consider two cases:

(1) Let $2k+2 \leq n \leq 3k+1$, then $S/(I(C_n^k) : x_{n-k}) = S/(I(P_n^k) : x_{n-k})$ so by the proof of Theorem 3.14 we have $\mathrm{sdepth}(S/(I(P_n^k) : x_{n-k})) = 2 = \lceil \frac{n}{2k+1} \rceil$. Therefore

$$\mathrm{sdepth}(S/I(C_n^k)) \leq \mathrm{sdepth}(S/(I(C_n^k) : x_{n-k})) = 2 = \lceil \frac{n}{2k+1} \rceil.$$

(2) Let $n \geq 3k+2$, then by Lemma 4.2

$$S/(I(C_n^k) : x_{n-k}) \cong S_{n-2k-1}/I(P_{n-2k-1}^k)[x_{n-k}].$$

By Lemma 3.7 and Theorem 3.14, $\mathrm{sdepth}(S_{n-2k-1}/I(P_{n-2k-1}^k)[x_{n-k}]) = \lceil \frac{n}{2k+1} \rceil$. Thus $\mathrm{sdepth}(S/I(C_n^k)) \leq \mathrm{sdepth}(S/(I(C_n^k) : x_{n-k})) = \lceil \frac{n}{2k+1} \rceil$.

□

5 Lower bounds for Stanley depth of edge ideals of k^{th} powers of paths and cycles and a conjecture of Herzog

In this section we compute some lower bounds for Stanley depth of $I(P_n^k)$ and $I(C_n^k)$. These bounds are good enough to prove that Conjecture 1.1 is true for $I(P_n^k)$ and $I(C_n^k)$. Let $0 \leq i \leq k-1$, define

$$R_{n-k+i} := K[\{x_1, x_2, \ldots, x_n\} \backslash \{x_{n-k}, x_{n-k+1}, \ldots, x_{n-k+i}\}]$$

and

$$B'_{n-k+i} := (x_j : x_j \in N_{P_n^k}(x_{n-k+i}) \backslash \{x_{n-k}, x_{n-k+1}, \dots, x_{n-k+(i-1)}\}).$$

Thus R_{n-k+i} is a subring of S and B'_{n-k+i} is a monomial prime ideal of S. Let $I \subset Z = K[x_{i_1}, x_{i_1}, \dots, x_{i_r}]$ be a monomial ideal and $Z' := Z[x_{i_r+1}]$. Then we write $IZ' = I[x_{i_r+1}]$. Now we recall a useful remark of Cimpoeas.

Remark 5.1. *[4, Remark 1.7]* Let I be a monomial ideal of S, and $I' = (I, x_{n+1}, x_{n+2}, \dots, x_{n+m})$ be a monomial ideal of $S' = S[x_{n+1}, x_{n+2}, \dots, x_{n+m}]$. Then

$$\operatorname{sdepth}_{S'}(I') \geq \min\{\operatorname{sdepth}_S(I) + m, \operatorname{sdepth}_S(S/I) + \lceil \frac{m}{2} \rceil\}.$$

Theorem 5.2. *Let $n \geq 2$, then* $\operatorname{sdepth}(I(P_n^k)) \geq \lceil \frac{n}{2k+1} \rceil + 1$.

Proof. (a) If $n \leq 2k+1$, then as the minimal generators of $I(P_n^k)$ have degree 2, by [15, Lemma 2.1] we have $\operatorname{sdepth}(I(P_n^k)) \geq 2 = \lceil \frac{n}{2k+1} \rceil + 1$.

(b) For $n \geq 2k+2$, if $k = 1$, then by [19, Theorem 2.3], $\operatorname{sdepth}(I(P_n^1)) \geq n - \lfloor \frac{n-1}{2} \rfloor = \lceil \frac{n-1}{2} \rceil + 1 \geq \lceil \frac{n}{3} \rceil + 1$. Now for $k \geq 2$, we prove this result by induction on n. We consider the following decomposition of $I(P_n^k)$ as a vector space:

$$I(P_n^k) = I(P_n^k) \cap R_{n-k} \oplus x_{n-k}(I(P_n^k) : x_{n-k})S.$$

Similarly, we can decompose $I(P_n^k) \cap R_{n-k}$ as follows:

$$I(P_n^k) \cap R_{n-k} = I(P_n^k) \cap R_{n-k+1} \oplus x_{n-k+1}(I(P_n^k) \cap R_{n-k} : x_{n-k+1})R_{n-k}.$$

Continuing in the same way for $1 \leq i \leq k-1$ we have

$$I(P_n^k) \cap R_{n-k+i} = I(P_n^k) \cap R_{n-k+(i+1)} \oplus \\ x_{n-k+(i+1)}(I(P_n^k) \cap R_{n-k+i} : x_{n-k+(i+1)})R_{n-k+i}.$$

Finally we get the following decomposition of $I(P_n^k)$:

$$I(P_n^k) = I(P_n^k) \cap R_{n-1} \oplus \\ \oplus_{i=1}^{k-1} x_{n-k+i}(I(P_n^k) \cap R_{n-k+(i-1)} : x_{n-k+i})R_{n-k+i} \oplus x_{n-k}(I(P_n^k) : x_{n-k})S.$$

Therefore

$$\operatorname{sdepth}(I(P_n^k)) \geq \min\big\{\operatorname{sdepth}(I(P_n^k) \cap R_{n-1}), \operatorname{sdepth}((I(P_n^k) : x_{n-k})S), \\ \min_{i=1}^{k-1}\{\operatorname{sdepth}((I(P_n^k) \cap R_{n-k+(i-1)} : x_{n-k+i})R_{n-k+i})\}\big\}.$$

As $I(P_n^k) \cap R_{n-1} = \mathcal{G}(I(P_{n-k-1}^k))[x_n]$, thus by induction on n and Lemma 3.7 we have $\operatorname{sdepth}(I(P_n^k) \cap R_{n-1}) \geq \lceil \frac{n-k-1}{2k+1} \rceil + 1 + 1 \geq \lceil \frac{n}{2k+1} \rceil + 1$. Now we need to show that $\operatorname{sdepth}((I(P_n^k) : x_{n-k})S) \geq \lceil \frac{n}{2k+1} \rceil + 1$ and

$$\operatorname{sdepth}((I(P_n^k) \cap R_{n-k+(i-1)} : x_{n-k+i})R_{n-k+i}) \geq \lceil \frac{n}{2k+1} \rceil + 1.$$

For this we consider the following cases:

(1) Let $2k+2 \leq n \leq 3k+1$. If $n = 2k+2$, then $(I(P_n^k) : x_{n-k})S = (x_2, \ldots, x_{n-k-1}, x_{n-k+1}, \ldots, x_n)S$, thus by [2, Theorem 2.2] and Lemma 3.7 we have

$$\operatorname{sdepth}((I(P_n^k) : x_{n-k})S) = \lceil \frac{n-2}{2} \rceil + 2 \geq \lceil \frac{n}{2k+1} \rceil + 1.$$

If $2k+3 \leq n \leq 3k+1$, then by Remark 3.6, we get

$$(I(P_n^k) : x_{n-k})S = (\mathcal{G}(I(P_{n-2k-1}^{f(n-k)})), B_{n-k})[x_{n-k}].$$

Since $\operatorname{sdepth}(I(P_{n-2k-1}^{f(n-k)})) + |\mathcal{G}(B_{n-k})| \geq 2$, by Remark 3.5 we have

$$\operatorname{sdepth}(S_{n-2k-1}/I(P_{n-2k-1}^{f(n-k)})) + \lceil \frac{|\mathcal{G}(B_{n-k})|}{2} \rceil \geq 2,$$

then by Remark 5.1, $\operatorname{sdepth}(\mathcal{G}(I(P_{n-2k-1}^{f(n-k)})), B_{n-k}) \geq 2$, and by Lemma 3.7 we have $\operatorname{sdepth}((I(P_n^k) : x_{n-k})S) \geq 3 = \lceil \frac{n}{2k+1} \rceil + 1$. Now since

$$(I(P_n^k) \cap R_{n-k+(i-1)} : x_{n-k+i})R_{n-k+i}) = (\mathcal{G}(I(P_{n-2k-1+i}^{f(n-k+i)})), B'_{n-k+i})[x_{n-k+i}].$$

So by the same arguments we have

$$\operatorname{sdepth}((I(P_n^k) \cap R_{n-k+(i-1)} : x_{n-k+i})R_{n-k+i}) \geq 3 = \lceil \frac{n}{2k+1} \rceil + 1.$$

(2) If $n \geq 3k+2$, then by the proof of Lemma 3.3 $(I(P_n^k) : x_{n-k})S = (\mathcal{G}(I(P_{n-2k-1}^k)), B_{n-k})[x_{n-k}]$ and

$$(I(P_n^k) \cap R_{n-k+(i-1)} : x_{n-k+i})R_{n-k+i} = (\mathcal{G}(I(P_{n-2k-1+i}^k)), B'_{n-k+i})[x_{n-k+i}].$$

By Remark 5.1 we have

$$\operatorname{sdepth}(\mathcal{G}(I(P^k_{n-2k-1})), B_{n-k}) \geq \min\Big\{ \operatorname{sdepth}(\mathcal{G}(I(P^k_{n-2k-1})))+ |\mathcal{G}(B_{n-k})|, \operatorname{sdepth}(S_{n-2k-1}/I(P^k_{n-2k-1})) + \lceil \frac{|\mathcal{G}(B_{n-k})|}{2} \rceil \Big\}.$$

By induction on n we have $\operatorname{sdepth}(\mathcal{G}(I(P^k_{n-2k-1}))) \geq \lceil \frac{n-2k-1}{2k+1} \rceil + 1 = \lceil \frac{n}{2k+1} \rceil$, and by Theorem 3.14, $\operatorname{sdepth}(S_{n-2k-1}/I(P^k_{n-2k-1})) = \lceil \frac{n}{2k+1} \rceil - 1$. Therefore $\operatorname{sdepth}(\mathcal{G}(I(P^k_{n-2k-1})), B_{n-k}) \geq \lceil \frac{n}{2k+1} \rceil + 1$. Thus by Lemma 3.7 we have $\operatorname{sdepth}((I(P^k_n) : x_{n-k})S) > \lceil \frac{n}{2k+1} \rceil + 1$. Now using Remark 5.1 again, we get

$$\begin{aligned}\operatorname{sdepth}(\mathcal{G}(I(P^k_{n-2k-1+i})), B'_{n-k+i}) \geq \\ \min\Big\{ \operatorname{sdepth}(\mathcal{G}(I(P^k_{n-2k-1+i}))) + |\mathcal{G}(B'_{n-k+i})|, \\ \operatorname{sdepth}(S_{n-2k-1+i}/I(P^k_{n-2k-1+i})) + \lceil \frac{|\mathcal{G}(B'_{n-k+i})|}{2} \rceil \Big\}.\end{aligned}$$

By induction on n we have $\operatorname{sdepth}(\mathcal{G}(I(P^k_{n-2k-1+i}))) \geq \lceil \frac{n-2k-1+i}{2k+1} \rceil + 1$, and by Theorem 3.14 we have $\operatorname{sdepth}(S_{n-2k-1+i}/I(P^k_{n-2k-1+i})) = \lceil \frac{n-2k-1+i}{2k+1} \rceil$. Therefore

$$\operatorname{sdepth}(\mathcal{G}(I(P^k_{n-2k-1+i})), B'_{n-k+i}) \geq \lceil \frac{n-2k-1+i}{2k+1} \rceil + 1.$$

Thus by Lemma 3.7

$$\operatorname{sdepth}((I(P^k_n) \cap R_{n-k+(i-1)} : x_{n-k+i})R_{n-k+i}) \geq \lceil \frac{n}{2k+1} \rceil + 1.$$

This completes the proof.

□

Proposition 5.3. *Let* $n \geq 2k+1$, *then* $\operatorname{sdepth}(I(C^k_n)/I(P^k_n)) \geq \lceil \frac{n+k+1}{2k+1} \rceil$.

Proof. When $k = 1$, then by [6, Proposition 1.10] we have the required result. Now assume that $k \geq 2$ and consider the following cases:

(1). If $2k+1 \leq n \leq 3k+1$, then as $I(C^k_n)$ is a monomial ideal generated by degree 2 so by [11, Theorem 2.1] $\operatorname{sdepth}(I(C^k_n)/I(P^k_n)) \geq 2 = \lceil \frac{n+k+1}{2k+1} \rceil$.

(2). If $3k+2 \leq n \leq 4k+1$, then we use [11] to show that there exist Stanley decompositions of desired Stanley depth. Let $s \in \{1,2,\dots k\}$, $j_s \in \{1,2,\dots,k+1-s\}$ and

$$L := \oplus_{s=1}^{k}\Big(\oplus_{j_s=1}^{k+1-s} x_{j_s}x_{n+1-s}K[x_{j_s}, x_{j_s+k+1}, x_{n+1-s}]\Big).$$

It is easy to see that $L \subset I(C_n^k)\backslash I(P_n^k)$. Now let $u_i \in I(C_n^k)\backslash I(P_n^k)$ be a squarefree monomial such that $u_i \notin L$ then clearly $\deg(u_i) \geq 3$. Since

$$I(C_n^k)/I(P_n^k) \cong L \oplus_{u_i} u_iK[\operatorname{supp}(u_i)]$$

Thus $\operatorname{sdepth}(I(C_n^k)/I(P_n^k)) \geq 3 = \lceil \frac{n+k+1}{2k+1} \rceil$ as required.

(3). If $n \geq 4k+2$, then we have the following K-vector space isomorphism:

$$I(C_n^k)/I(P_n^k) \cong$$

$$\oplus_{j_1=1}^{k} x_{j_1}x_n \frac{K[x_{j_1+k+1}, x_{j_1+k+2}, \dots, x_{n-k-1}]}{(x_{j_1+k+1}x_{j_1+k+2}, x_{j_1+k+1}x_{j_1+k+3}, \dots, x_{n-k-2}x_{n-k-1})}[x_{j_1}, x_n]\oplus$$

$$\oplus_{j_2=1}^{k-1} x_{j_2}x_{n-1} \frac{K[x_{j_2+k+1}, x_{j_2+k+2}, \dots, x_{n-k-2}]}{(x_{j_2+k+1}x_{j_2+k+2}, x_{j_2+k+1}x_{j_1+k+3}, \dots, x_{n-k-3}x_{n-k-2})}[x_{j_2}, x_{n-1}]\oplus$$

$$\vdots$$

$$\oplus_{j_{k-1}=1}^{2} x_{j_{k-1}}x_{n-k+2} \frac{K[x_{j_{k-1}+k+1}, x_{j_{k-1}+k+2}, \dots, x_{n-2k+1}]}{(x_{j_{k-1}+k+1}x_{j_{k-1}+k+2}, \dots, x_{n-2k}x_{n-2k+1})}[x_{j_{k-1}}, x_{n-k+2}]\oplus$$

$$x_1x_{n-(k-1)} \frac{K[x_{k+2}, x_{k+3}, \dots, x_{n-2k}]}{(x_{k+2}x_{k+3}, x_{k+2}x_{k+4}, \dots, x_{n-2k-1}x_{n-2k})}[x_1, x_{n-(k-1)}].$$

Thus

$$I(C_n^k)/I(P_n^k) \cong$$

$$\oplus_{s=1}^{k}\Big(\oplus_{j_s=1}^{k+1-s} x_{j_s}x_{n+1-s}\big(S_{j_s+k+1,n-s-k}/\big(\mathcal{G}(I(P_n^k))\cap S_{j_s+k+1,n-s-k}\big)[x_{j_s}, x_{n+1-s}]\Big),$$

where $S_{j_s+k+1,n-s-k} = K[x_{j_s+k+1}, x_{j_s+k+2}, \dots, x_{n-s-k}]$. Indeed, if $u \in I(C_n^k)$ such that $u \notin I(P_n^k)$ then $(x_{j_s}x_{n+1-s})|u$ for only one pair of s and j_s. If $(x_{j_s}x_{n+1-s})|u$ then $u = x_{j_s}^{\gamma}x_{n+1-s}^{\delta}v$ and $v \in S_{j_s+k+1,n-s-k}$. Since $v \notin I(P_n^k)$, it follows that $v \notin \mathcal{G}(I(P_n^k)) \cap S_{j_s+k+1,n-s-k}$. Clearly

$$S_{j_s+k+1,n-s-k}/\mathcal{G}(I(P_n^k)) \cap S_{j_s+k+1,n-s-k} \cong S_{n-(j_s+2k+s)}/I(P^k_{n-(j_s+2k+s)}).$$

Thus by Theorem 3.14 and Lemma 3.7, we have

$$\operatorname{sdepth}(I(C_n^k)/I(P_n^k)) \geq \min_{s=1}^{k}\{\lceil \frac{n-(j_s+s+2k)}{2k+1} \rceil + 2\}.$$

It is easy to see that $\max\{j_s + s\} = k + 1$. Therefore

$$\operatorname{sdepth}(I(C_n^k)/I(P_n^k)) \geq \lceil \frac{n-(3k+1)}{2k+1} \rceil + 2 = \lceil \frac{n+k+1}{2k+1} \rceil.$$

□

Theorem 5.4. *Let $n \geq 3$, then*

$$\begin{aligned} &\operatorname{sdepth}(I(C_n^k)) \geq 2, && \text{if } n \leq 2k+1;\\ &\operatorname{sdepth}(I(C_n^k)) \geq \lceil \frac{n-k}{2k+1} \rceil + 1, && \text{if } n \geq 2k+2. \end{aligned}$$

Proof. (a) If $n \leq 2k+1$, then as the minimal generators of $I(C_n^k)$ have degree 2, so by [15, Lemma 2.1] $\operatorname{sdepth}(I(C_n^k)) \geq 2$.

(b) If $n \geq 2k+2$, then consider the short exact sequence

$$0 \longrightarrow I(P_n^k) \longrightarrow I(C_n^k) \longrightarrow I(C_n^k)/I(P_n^k) \longrightarrow 0,$$

by Lemma 2.4 we have

$$\operatorname{sdepth}(I(C_n^k)) \geq \min\{\operatorname{sdepth}(I(P_n^k)), \operatorname{sdepth}(I(C_n^k)/I(P_n^k))\}.$$

By Theorem 5.2, $\operatorname{sdepth}(I(P_n^k)) \geq \lceil \frac{n}{2k+1} \rceil + 1$, and by Proposition 5.3, we obtain $\operatorname{sdepth}(I(C_n^k)/I(P_n^k)) \geq \lceil \frac{n+k+1}{2k+1} \rceil = \lceil \frac{n-k}{2k+1} \rceil + 1$.

□

Corollary 5.5. *Let $n \geq 3$, if $n \leq 2k+1$, then* $\operatorname{sdepth}(I(C_n^k)) \geq 2 = \operatorname{sdepth}(S/I(C_n^k)) + 1$. *If $n \geq 2k+2$, then*

$$\begin{aligned} &\operatorname{sdepth}(I(C_n^k)) \geq \operatorname{sdepth}(S/I(C_n^k)), && \text{if } n \equiv 1, \dots, k \,(\operatorname{mod}(2k+1));\\ &\operatorname{sdepth}(I(C_n^k)) \geq \operatorname{sdepth}(S/I(C_n^k)) + 1, && \text{if } n \equiv 0, k+1, \dots, 2k \,(\operatorname{mod}(2k+1)). \end{aligned}$$

Proof. Proof follows by Corollary 4.8, Theorem 4.7 and Theorem 5.4. □

Acknowledgement

The authors would like to thank the referee for a careful reading of the paper and for valuable comments. This research is partially supported by HEC Pakistan.

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